

# $(l, q)$ -Deformed Grassmann Field and the Two-dimensional Ising Model

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## Abstract

In this paper we construct the exact representation of the Ising partition function in the form of the  $SL_q(2, R)$ -invariant functional integral for the lattice free  $(l, q)$ -fermion field theory ( $l = q = -1$ ). It is shown that the  $(l, q)$ -fermionization allows one to re-express the partition function of the eight-vertex model in external field through functional integral with four-fermion interaction. To construct these representations, we define a lattice  $(l, q, s)$ -deformed Grassmann bispinor field and extend the Berezin integration rules to this field. At  $l = q = -1, s = 1$  we obtain the lattice  $(l, q)$ -fermion field which allows us to fermionize the two-dimensional Ising model. We show that the Gaussian integral over  $(q, s)$ -Grassmann variables is expressed through the  $(q, s)$ -deformed Pfaffian which is equal to square root of the determinant of some matrix at  $q = \pm 1, s = \pm 1$ .

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# 1 Introduction

During the last decade, a considerable progress in understanding the meaning of the infinite-dimensional dynamical symmetries in the exactly solvable two-dimensional lattice spin models and quantum field theories has been achieved. As it was shown by Belavin, Polyakov and Zamolodchikov in [1], conformal invariance of the critical fluctuations in the two-dimensional lattice spin systems allows one to find a universality classes classification of the critical behaviour of exactly solvable lattice systems and to assign to each class the corresponding two-dimensional quantum conformal field theory. This result is connected with the fact that, at critical points, such systems, besides the invariance with respect to the global conformal transformations ( $SL(2, C)$  group) possess larger symmetry, that is, the space of eigenvectors of the transfer matrix (Fock space for quantum conformal field theories) has an invariance with respect to the infinite-dimensional Virasoro algebra.

However, as soon as the correlation radius of critical fluctuations is finite nearby the critical point, the conformal invariance is broken. Nevertheless, as it was shown by Zamolodchikov in [2], exactly solvable two-dimensional quantum field theory can still possess infinitely many integrals of motion, in the vicinity of the fixed point (this point corresponds to a quantum conformal field theory). This allowed him to suggest that the Fock space could be realized as an irreducible representation space for some infinite-dimensional algebra. This idea has been explicitly realized for exactly solvable massive quantum field theories. For example, in the case of quantum Sine-Gordon model, the Fock space has  $U_q(\hat{sl}_2)$  symmetry [3,4] and the deformation parameter  $q$  is connected with the coupling constant of the model. Also, as it was shown in [3,4] for the  $SU(2)$ -invariant Thirring model, the creation and annihilation operators realize a representation of  $U_q(\hat{sl}_2)$  at  $q = -1$ .

In [6] there have been also found infinitely many integrals of motion nearby the critical point of the eight-vertex model, corresponding to the two non-interacting Ising sub-lattices. In the remarkable series of papers by Jimbo, Miwa and their collaborators [7,8], it was shown that the eigenvectors of the corner transfer-matrix for the six-vertex model form the irreducible representation space of  $U_q(\hat{sl}_2)$  at  $-1 < q < 0$ . Moreover, it was demonstrated in [9,10] that the space of eigenvectors of the corner transfer-matrix for the RSOS models and, in particular, for the two-dimensional Ising model [10] can possess this symmetry too.

These results allow one to assume that the partition function of the two-dimensional exactly solvable lattice model can be expressed through the partition function of some  $SL_q(2, C)$ -invariant lattice  $q$ -deformed field theory.

In this paper, we make the first step examining this assumption and propose a simple example of the construction of such a type (lattice field theory) in the case of the two-dimensional Ising model, making use of the exact representation of the Ising partition function as the  $SL_q(2, R)$ -invariant functional integral in the lattice free  $(l, q)$ -fermion field theory ( $l = q = -1$ ). Moreover, it is shown that the method of  $q$ -fermionization allows one to re-express the partition function of the eight-vertex model in external field through the functional integral with four-fermion interaction.

For the construction of these representations we define a lattice  $(l, q, s)$ -deformed Grassmann bispinor field<sup>2</sup> and extend the Berezin integration rules to this field. Here  $l$  is a de-

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<sup>2</sup>Note that for  $l = e^{i\pi\theta}$  the lattice  $(l, q)$ -deformed field one can consider as  $q$ -deformed non-abelian anyonic field

formation parameter for the commutation relations of "values" of this field in two arbitrary lattice sites,  $q$  is a deformation parameter for  $q$ -Grassmann spinor and  $s$  is a deformation parameter for commutation relations of two  $q$ -Grassmann spinors in a  $s$ -deformed bispinor. At  $l = q = -1, s = 1$  we obtain the lattice  $(l, q)$ -deformed field which allows us to fermionize the two-dimensional Ising model (we call it the lattice  $(l, q)$ -fermion field). Moreover, we show that Gaussian integral over  $(q, s)$ -Grassmann variables is expressed through the  $(q, s)$ -deformed Pfaffian which at  $q = \pm 1, s = \pm 1$  is equal to square root of the determinant of some matrix.

It is a standard point that, in the scaling limit  $T \rightarrow T_c, a \rightarrow 0$ , the partition function of the two-dimensional Ising model is considered as the partition function of the free massive Majorana fermion theory [11]. One can obtain this result by representing the Ising partition function as the functional integral in the lattice real fermion field theory [12-16].

The fermionization method proposed in [13,15] proved to be effective for solving to the boundary condition problems and allows one to calculate the partition function of the two-dimensional Ising model with an arbitrary magnetic field attached to the boundaries [16].

Now we briefly recall the results obtained in [13,15]. Partition function of the Ising model on the torus is

$$Z_I = \sum_{\{\sigma\}} e^{-\beta H} = \sum_{\{\sigma\}} \exp \left[ \mathcal{K} \sum_r (\sigma_{r+\hat{x}} \sigma_r + \sigma_{r+\hat{y}} \sigma_r) \right], \quad (1.1)$$

where  $r = (x, y)$  denotes the sites of the square lattice of size  $n \times m$ ,  $x = 1, \dots, n$ ,  $y = 1, \dots, m$ ,  $\mathcal{K} = \beta J$ ,  $\sigma_r = \pm 1$ , and  $\hat{x}, \hat{y}$  are unit vectors along the horizontal  $X$  and vertical  $Y$  axes respectively. This partition function can be presented as the sum of the functional integrals over the lattice real fermion field

$$Z_I = Z^{AA} + Z^{AP} + Z^{PA} - Z^{PP}, \quad (1.2)$$

$$Z = \text{const} \int \mathcal{D}\psi \exp \left[ \frac{1}{2} \sum \psi_r^i D_{r,r'}^{ij} \psi_{r'}^j \right], \quad (1.3)$$

where

$$\hat{D} = \begin{bmatrix} 0 & 1 & 1+t\nabla_x & 1 \\ -1 & 0 & 1 & 1+t\nabla_y \\ -1-t\nabla_{-x} & -1 & 0 & 1 \\ -1 & -1-t\nabla_{-y} & -1 & 0 \end{bmatrix} \quad (1.4)$$

$$\nabla_x \psi_r = \psi_{r+\hat{x}}, \quad \nabla_y \psi_r = \psi_{r+\hat{y}}, \quad t = \tanh \mathcal{K}, \quad \mathcal{D}\psi = \prod_{i,r} d\psi_r^i.$$

Here, for example, the indices  $P, A$  in  $Z^{PA}$  denote periodic and antiperiodic boundary conditions for the field  $\psi_r$  along the horizontal  $X$  and vertical  $Y$  axes respectively. The appearance of the four-component Grassmann (bispinor) field  $\psi_r^\alpha$  is related to the requirement of linearity of the quadratic form in (1.3) with respect to the shift operators  $\nabla_x, \nabla_y$ . Note also that for such a quadratic form the doubling problem for the lattice fermions, arising in the course of usual formulations of lattice fermion field theories [17] is absent.

In the momentum representation, the matrix  $\hat{D}$  has the following form

$$D^{ij}(p_1, p_2) = \begin{bmatrix} 0 & 1 & 1 + te^{ip_1} & 1 \\ -1 & 0 & 1 & 1 + te^{ip_2} \\ -1 - te^{-ip_1} & -1 & 0 & 1 \\ -1 & -1 - te^{-ip_2} & -1 & 0 \end{bmatrix}, \quad (1.5)$$

where for the periodic and antiperiodic boundary conditions along the  $X, Y$  axes,  $p_1, p_2$  are integer and half-integer respectively.

Using momentum representation, it is not hard to calculate functional integral (1.3) and, for example, one can obtain for the  $Z^{AA}$

$$Z^{AA} = \text{const} \sqrt{\det \hat{D}} = c \prod_{p_1, p_2} (s + s^{-1} - \cos p_1 - \cos p_2), \quad (1.6)$$

where  $s = \sinh 2\mathcal{K}$  and  $c = \text{const}$ ,  $\sinh 2\mathcal{K}_c = 1$  at  $T = T_c$ .

Introducing the notation

$$\cosh \gamma(p_2) = s + s^{-1} - \cos p_2 \quad (1.7)$$

and calculating the product over  $p_1$ , one gets

$$Z^{AA} = c \prod_p \cosh \frac{m}{2} \gamma(p) = c \prod_p \exp\left(\frac{1}{2} m a \frac{\gamma(p)}{a}\right) (1 + \exp(-m a \frac{\gamma(p)}{a})). \quad (1.8)$$

Here  $a$  is a dimensional lattice spacing, and we define the dimensional momentum by substitution  $p_2 \rightarrow pa$ . In order to study the continuum limit, one expands (1.7) into a power series in  $p$  around  $p = 0$ . Restricting ourselves to the quadratic terms in  $p$ , we obtain

$$\frac{\gamma^2(p)}{a^2} = \frac{2(s-1)^2}{sa^2} + p^2 = \varepsilon^2(p). \quad (1.9)$$

Considering different limits in (1.9), one gets  $\varepsilon(p)$  either for the massless or for the massive Majorana fermions:

$$1). T = T_c, \quad s = 1, \quad a \rightarrow 0,$$

$$\mu = \lim_{\substack{s \rightarrow 1 \\ a \rightarrow 0}} \sqrt{\frac{2s-1}{s}} \frac{1}{a} = 0 \quad \text{and} \quad \varepsilon^2(p) = p^2,$$

$$2). T \rightarrow T_c, \quad s \rightarrow 1, \quad a \rightarrow 0,$$

$$\mu = \lim_{\substack{s \rightarrow 1 \\ a \rightarrow 0}} \sqrt{\frac{2s-1}{s}} \frac{1}{a} = \sqrt{2} \quad \text{and} \quad \varepsilon^2(p) = \mu^2 + p^2.$$

The free energy of the Ising model in the continuum limit turns into the free energy of Majorana fermions at finite temperature. To see this, let us define, as  $a \rightarrow 0$ ,  $n \rightarrow \infty$ ,  $m \rightarrow$

$\infty$ , the system volume as  $L = na$  and the inverse temperature of the thermostat in which Majorana fermions are placed as  $\tilde{\beta} = \tilde{T}^{-1} = ma$ . It is not hard to show that in the thermodynamic limit ( $L \rightarrow \infty$ ) the main contribution to the free energy comes from  $Z^{AA}$ . Then, for the free energy density  $f$  one obtains from (1.8)

$$f = -\beta^{-1} \int_{-\infty}^{\infty} dp \ln[1 + \exp(-\tilde{\beta}\varepsilon(p))].$$

This expression confirms ("+" in front of the exponential) that the partition function of the two-dimensional Ising model on the strip of width  $\tilde{\beta} = ma$  in the continuum limit describes the Majorana fermion system in the thermostat.

This paper is organized as follows. In section 2 we give a short introduction to  $q$ -spinors and the quantum matrix group  $SL_q(2, C)$ . In section 3 we define the lattice  $(l, q, s)$ -deformed Grassmann bispinor field. In section 4 we define integral calculus for this field. In section 5 we propose a method to represent the partition function of the two-dimensional Ising model with the nearest-, next-nearest-neighbour and four-spin interactions (eight-vertex model in external field) in the form of the functional integral over the lattice  $(l, q)$ -fermion field ( $l = q = -1, s = 1$ ). To realize this representation we use the  $GL_q(2, C)$ -covariant generalization of the Berezin integration over Grassmann field proposed in section 4 and calculate the functional integral for the Ising model with the nearest-neighbour interaction.

## 2 $q$ -Spinors and the quantum group $SL_q(2, C)$

Let us briefly recall the known facts about  $q$ -spinors, their  $q$ -Grassmann realization and the quantum matrix group  $SL_q(2, C)$  [18-23]. By  $q$ -spinor  $v$  we mean the two-component object

$$v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix},$$

with the following commutation relation for the components

$$v^2 v^1 = \frac{1}{q} v^1 v^2, \quad (2.1)$$

where  $q$  is generally a complex number. At  $q = 1$  it is usual spinor. Following the paper [19], one can look at  $q$ -spinors as belonging to the quantum plane  $A_q^{2|0}$ .

One can consider this quantum plane as the two-dimensional representation space for the quantum matrix group  $SL_q(2, C)$  ( $q$ -deformed  $SL(2, C)$ ) [18-23]. Using the  $\hat{R}$ -matrix for this group, one can rewrite commutation relation (2.1) in the covariant form

$$v^\alpha v^\beta = \frac{1}{q} \hat{R}_{q\gamma\rho}^{\alpha\beta} v^\gamma v^\rho, \quad (2.2)$$

where Greek indices go over 1, 2 and  $\hat{R}_q$  is a symmetric  $4 \times 4$  matrix (the rows and columns are numbered as (11), (12), (21), (22))

$$\hat{R}_{q\gamma\rho}^{\alpha\beta} = \hat{R}_{q\alpha\beta}^{\gamma\rho}, \quad \hat{R}_q = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \quad (2.3)$$

This matrix has eigenvalues  $q$  and  $-q^{-1}$  and satisfies the quantum Yang-Baxter equation [18-20]

$$\hat{R}_{q12} \hat{R}_{q23} \hat{R}_{q12} = \hat{R}_{q23} \hat{R}_{q13} \hat{R}_{q23}, \quad (2.4)$$

and the Hecke relation

$$\hat{R}_q^2 - (q - q^{-1})\hat{R}_q - \hat{I} = (\hat{I} + q\hat{R}_q)(\hat{I} - q^{-1}\hat{R}_q) = 0, \quad (2.5)$$

where  $\hat{I} = I_{\gamma\rho}^{\alpha\beta} = \delta_\gamma^\alpha \delta_\rho^\beta$  is the unit matrix. One can write the following representation for the  $\hat{R}$ -matrix [20-22]

$$\hat{R}_q = q\hat{S}_q - q^{-1}\hat{A}_q,$$

where matrices

$$\hat{A}_q = \frac{1}{1 + q^{-2}}(\hat{I} - q^{-1}\hat{R}_q), \quad (2.6)$$

$$\hat{S}_q = \frac{1}{1 + q^2}(\hat{I} + q\hat{R}_q) \quad (2.7)$$

are orthonormal projectors (to the subspaces with eigenvalues  $-q^{-1}$  and  $q$  respectively)

$$\hat{A}_q \cdot \hat{S}_q = \hat{S}_q \cdot \hat{A}_q = 0, \quad \hat{A}_q^2 = \hat{A}_q, \quad \hat{S}_q^2 = \hat{S}_q.$$

These projections are the quantum counterparts of the classical ( $q = 1$ ) antisymmetrizer and symmetrizer for the tensors with two indices.

Using the  $q$ -antisymmetrizer  $\hat{A}_q$ , one can rewrite commutation relations (2.2)

$$\hat{A}_q^{\alpha\beta}{}_{\gamma\rho} v^\gamma v^\rho = 0.$$

The classical ( $q = 1$ ) Grassmann spinor  $z^\alpha$  can be defined through

$$\hat{S}_q^{\alpha\beta}{}_{\gamma\rho} z^\gamma z^\rho = 0,$$

or, in components,

$$(z^1)^2 = (z^2)^2 = 0, \quad z^1 z^2 + z^2 z^1 = 0.$$

Analogously, one can define the  $q$ -Grassmann spinor  $\psi$  at  $q^2 \neq -1$  [21,22]

$$\hat{S}_q^{\alpha\beta}{}_{\gamma\rho} \psi^\gamma \psi^\rho = 0, \quad \psi^\alpha \psi^\beta = -q \hat{R}_{q\gamma\rho}^{\alpha\beta} \psi^\gamma \psi^\rho, \quad (2.8)$$

or, in components,

$$(\psi^1)^2 = (\psi^2)^2 = 0, \quad \psi^2 \psi^1 = -q \psi^1 \psi^2. \quad (2.9)$$

It implies (see the paper [19]) that  $\psi$  belongs to the quantum plane  $A_q^{0|2}$ , which we denote as  $\Psi^{(2)}$ .

In order to show that commutation relation (2.8) is invariant with respect to the transformation of  $\psi$  by a quantum matrix  $\hat{A} \in SL_q(2, C)$

$$\tilde{\psi}^\alpha = \hat{A}_\beta^\alpha \psi^\beta = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \quad (2.10)$$

we recall some properties of  $SL_q(2, C)$ .

The matrix elements  $\hat{A}_\beta^\alpha$  commute with the components  $\psi^\alpha$  and belong to the associative algebra of functions on the quantum group  $SL_q(2)$ , which we denote as  $\mathcal{A}$  or  $Fun_q(SL(2))$  [20]. This algebra is a Hopf algebra. It implies that the following maps are defined in this algebra:

a) comultiplication  $\Delta$

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} : \quad \Delta(A_\beta^\alpha) = A_\gamma^\alpha \otimes A_\beta^\gamma,$$

where symbol " $\otimes$ " denotes the tensor product of the quantum spaces,

b) counit  $\varepsilon$

$$\mathcal{A} \xrightarrow{\varepsilon} \mathbf{C} : \quad \varepsilon(A_\beta^\alpha) = \delta_\beta^\alpha,$$

where  $\mathbf{C}$  is a complex number,

c) antipode  $i$

$$\mathcal{A} \xrightarrow{i} \mathcal{A} : \quad i(A_\beta^\alpha) = (-q)^{\alpha-\beta} \tilde{A}_\alpha^\beta,$$

where  $\tilde{A}_\alpha^\beta$  is the quantum minor of the matrix element  $A_\beta^\alpha$  defined for the quantum matrix  $\hat{A} \in SL_q(2, C)$  in the following way [20]

$$\tilde{A}_1^1 = A_2^2, \quad \tilde{A}_2^2 = A_1^1, \quad \tilde{A}_1^2 = A_2^1, \quad \tilde{A}_2^1 = A_1^2,$$

d) multiplication  $m$

$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} : \quad m(A_\beta^\alpha \otimes A_\rho^\gamma) = A_b^\alpha A_\rho^\gamma,$$

and the matrix elements  $A_\beta^\alpha$  obey the commutation relations

$$\hat{R}_{q\gamma\rho}^{\alpha\beta} A_\mu^\gamma A_\nu^\rho = A_\gamma^\alpha A_\rho^\beta \hat{R}_{q\mu\nu}^{\gamma\rho}. \quad (2.11)$$

which are equivalent, due to (2.10), to

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \quad ad - da = (q - q^{-1})bc.$$

These maps allows one to define the following operations with quantum the matrix  $\hat{A} \in SL_q(2, C)$ :

a)  $\Delta(\hat{A}) = \hat{A} \hat{\otimes} \hat{A}$  defines the rule of multiplication of quantum matrices, and matrix elements of  $\hat{A} \hat{\otimes} \hat{A}$  have the form  $A_\gamma^\alpha \otimes A_\beta^\gamma$  ( $\hat{\otimes}$  denotes the tensor product along with the usual matrix multiplication),

b)  $\varepsilon(\hat{A}) = \hat{\mathbf{1}}$  defines the unit matrix in  $SL_q(2, C)$ ,  $\hat{\mathbf{1}} = \delta_\beta^\alpha$ ,

c)  $i(\hat{A}) = \hat{A}^{-1}$  defines the inverse matrix, that is  $i(\hat{A}) \cdot \hat{A} = \hat{A} \cdot i(\hat{A}) = \hat{\mathbf{1}}$ , where " $\cdot$ " denotes the matrix multiplication,

d)  $m(\hat{A} \otimes \hat{A}) = \hat{A} \hat{\otimes} \hat{A}$  defines the usual tensor product for quantum matrices, and matrix elements of  $\hat{A} \hat{\otimes} \hat{A}$  have the form  $A_\beta^\alpha A_\rho^\gamma$ .

e) the quantum determinant of the quantum matrix  $\hat{A}$

$$\det_q \hat{A} = \sum_{\alpha, \beta} (-q)^{\beta-\alpha} A_\beta^\alpha \tilde{A}_\alpha^\beta = 1.$$

Using (2.11), one immediately shows that it commutes with all the matrix elements  $A_\beta^\alpha$ . Linear transformation (2.10) of the quantum plane  $\Psi^{(2)}$  can be rewritten through the map  $\delta$

$$\Psi^{(2)} \xrightarrow{\delta} \hat{A} \dot{\otimes} \Psi^{(2)} : \quad \delta(\psi) = \hat{A} \dot{\otimes} \psi, \quad \delta(\psi^\alpha) = A_\beta^\alpha \otimes \psi^\beta.$$

It means that  $\Psi^{(2)}$  is a comodule of  $SL_q(2, C)$ . Then, using the map  $\delta \hat{\otimes} \delta$ , the action of quantum matrix  $\hat{A}$  on the tensor product  $\Psi^{(2)} \otimes \Psi^{(2)}$  can be defined as the action of the tensor product  $\hat{A} \hat{\otimes} \hat{A}$

$$\Psi^{(2)} \otimes \Psi^{(2)} \xrightarrow{\delta \hat{\otimes} \delta} (\hat{A} \dot{\otimes} \Psi^{(2)}) \otimes (\hat{A} \dot{\otimes} \Psi^{(2)}).$$

It is convenient to define the matrix  $\hat{A} \hat{\otimes} \hat{A}$  in the following way

$$\hat{A} \hat{\otimes} \hat{A} = A_1 \cdot A_2, \quad A_1 = \hat{A} \hat{\otimes} \hat{\mathbf{1}} = A_\beta^\alpha \delta_\rho^\gamma, \quad A_2 = \hat{\mathbf{1}} \hat{\otimes} \hat{A} = \delta_\beta^\alpha A_\rho^\gamma. \quad (2.12)$$

Hence, its action on  $\Psi^{(2)} \otimes \Psi^{(2)}$  can be written as

$$\begin{aligned} (\hat{A} \hat{\otimes} \hat{A}) \dot{\otimes} (\psi \otimes \psi) &= A_1 \cdot A_2 \dot{\otimes} (\psi \otimes \psi) = \\ (\hat{A} \hat{\otimes} \hat{\mathbf{1}}) \cdot (\hat{\mathbf{1}} \hat{\otimes} \hat{A}) \dot{\otimes} (\psi \otimes \psi) &= (\hat{A} \dot{\otimes} \psi) \otimes (\hat{A} \dot{\otimes} \psi) = \tilde{\psi} \otimes \tilde{\psi}. \end{aligned} \quad (2.13)$$

After using the matrices  $A_1$  and  $A_2$ , commutation relation (2.11) can be represented in the form

$$\hat{R}_q \cdot A_1 \cdot A_2 = A_1 \cdot A_2 \cdot \hat{R}_q. \quad (2.14)$$

Let us show that quantum matrices in (2.10), (2.11) form the quantum matrix group  $SL_q(2, C)$  with respect to the multiplication  $\dot{\otimes}$ . It is easy to show that matrix elements of  $\hat{A}'' = \hat{A}' \dot{\otimes} \hat{A}$  satisfy commutation relation (2.11), if  $\hat{A}', \hat{A} \in SL_q(2, C)$ . Indeed, one defines matrices

$$A'_1 = \hat{A}' \hat{\otimes} \hat{\mathbf{1}}, \quad A'_2 = \hat{\mathbf{1}} \hat{\otimes} \hat{A}',$$

which satisfy the relations

$$\hat{R}_q \cdot A'_1 \cdot A'_2 = A'_1 \cdot A'_2 \cdot \hat{R}_q, \quad A''_1 = A'_1 \dot{\otimes} A_1, \quad A''_2 = A'_2 \dot{\otimes} A_2. \quad (2.15)$$

Using the comultiplication property

$$(A'_1 \otimes A_1) \cdot (A'_2 \otimes A_2) = (A'_1 \cdot A'_2) \dot{\otimes} (A_1 \cdot A_2)$$

and (2.14), (2.15), we get

$$\begin{aligned} \hat{R}_q \cdot A''_1 \cdot A''_2 &= \hat{R}_q \cdot (A'_1 \dot{\otimes} A_1) \cdot (A'_2 \dot{\otimes} A_2) = \hat{R}_q \cdot (A'_1 \cdot A'_2) \dot{\otimes} (A_1 \cdot A_2) = \\ (A'_1 \cdot A'_2) \dot{\otimes} (A_1 \cdot A_2) \cdot \hat{R}_q &= (A'_1 \dot{\otimes} A_1) \cdot (A'_2 \dot{\otimes} A_2) \cdot \hat{R}_q = A''_1 \cdot A''_2 \cdot \hat{R}_q. \end{aligned}$$

Hence, one concludes that the matrix elements of  $\hat{A}''$  do really satisfy relation (2.11). Moreover, the  $q$ -determinant has the property

$$\det_q \hat{A}'' = \det_q (\hat{A}' \dot{\otimes} \hat{A}) = \det_q (\hat{A}') \otimes \det_q (\hat{A}).$$

Then, the matrix elements  $A''_\beta^\alpha \in Fun_q(SL(2))$ .



Now one can easily show the invariance of commutation relation (2.8) under transformation (2.10). To see this, it is convenient to rewrite (2.8) in the form

$$(\hat{I} + q\hat{R}_q) \cdot (\psi \otimes \psi) = 0.$$

Then, using (2.13) and (2.14), one proves the invariance by means of the following chain of equalities

$$\begin{aligned} (\hat{I} + q\hat{R}_q) \cdot (\tilde{\psi} \otimes \tilde{\psi}) &= (\hat{I} + q\hat{R}_q) \cdot (\hat{A} \dot{\otimes} \psi) \otimes (\hat{A} \dot{\otimes} \psi) = \\ (\hat{I} + q\hat{R}_q) \cdot A_1 \cdot A_2 \dot{\otimes} (\psi \otimes \psi) &= A_1 \cdot A_2 \cdot (\hat{I} + q\hat{R}_q) \dot{\otimes} (\psi \otimes \psi) = \\ (\hat{A} \hat{\otimes} \hat{A}) \dot{\otimes} (\hat{I} + q\hat{R}_q) \cdot (\psi \otimes \psi) &= 0. \end{aligned} \quad (2.16)$$

### 3 Lattice (1,q,s)-Grassmann bispinor field

In this section we define the lattice (1,q,s)-deformed Grassmann bispinor field which is needed for the  $(l, q)$ -fermionization of the two-dimensional Ising model. For definition of this field we use the results of the paper [24]. Recall that  $l$  is a deformation parameter of the commutation relations of this field at two different lattice sites,  $q$  is a deformation parameter for  $q$ -Grassmann spinor and  $s$  is a deformation parameter for the commutation relations of two  $q$ -Grassmann spinors in  $s$ -deformed bispinor.

At first let us define  $(q, s)$ -Grassmann bispinor  $\psi = \{\psi_i^\alpha\}$  ( $\alpha = 1, 2$  and  $i = 1, 2$ )

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} \psi_1^1 \\ \psi_1^2 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} \psi_2^1 \\ \psi_2^2 \end{pmatrix}, \quad (3.1)$$

where  $\psi$  belongs to a quantum vector space  $\Psi^{(b)}$  and  $\psi_1 \in \Psi_1^{(2)}$ ,  $\psi_2 \in \Psi_2^{(2)}$ . Then, from (2.8) it follows that  $\psi_1$  and  $\psi_2$  satisfy the commutation relations

$$\psi_i^\alpha \otimes \psi_i^\beta = -q \hat{R}_{q\gamma\rho}^{\alpha\beta} \psi_i^\gamma \otimes \psi_i^\rho. \quad (3.2)$$

We consider  $\Psi^{(b)}$  as the Hecke sum with respect to the matrix  $\hat{Q}_q$  [24]

$$\Psi^{(b)} = \Psi_1^{(2)} \oplus_{\hat{Q}} \Psi_2^{(2)}. \quad (3.3)$$

This matrix defines commutation relations of  $\psi_1$  and  $\psi_2$  in  $s$ -deformed bispinor  $\psi$ . Following definition of the Hecke sum [25,26], let us consider the  $\hat{Q}_q$ -matrix as linear map

$$Q_q : \Psi_1^{(2)} \otimes \Psi_2^{(2)} \longrightarrow \Psi_2^{(2)} \otimes \Psi_1^{(2)} :$$

$$Q_q(\psi_1^\alpha \otimes \psi_2^\beta) \equiv \hat{Q}_{q\gamma\rho}^{\alpha\beta} \psi_1^\gamma \otimes \psi_2^\rho = -\frac{1}{s} \psi_2^\alpha \otimes \psi_1^\beta, \quad (3.4)$$

$$Q_q^{-1}(\psi_2^\alpha \otimes \psi_1^\beta) \equiv (\hat{Q}_q^{-1})_{\gamma\rho}^{\alpha\beta} \psi_2^\gamma \otimes \psi_1^\rho = -s \psi_1^\alpha \otimes \psi_2^\beta,$$

where in general case the deformation parameter  $s$  is a complex number.

Following (3.2) and (3.4), we define the  $\hat{R}$ -matrix for  $\Psi^{(b)}$  as linear map  $\hat{R} : \Psi^{(b)} \otimes \Psi^{(b)} \longrightarrow \Psi^{(b)} \otimes \Psi^{(b)}$ :

$$\hat{R}(\psi_i^\alpha \otimes \psi_j^\beta) \equiv \hat{R}_{\gamma\rho ij}^{\alpha\beta kl} \psi_k^\gamma \otimes \psi_l^\rho = -\mathcal{I}_{s\gamma\rho ij}^{\alpha\beta kl} \psi_k^\gamma \otimes \psi_l^\rho, \quad (3.5)$$

where  $\hat{\mathcal{I}}_s$  is a diagonal matrix

$$\mathcal{I}_{s\gamma\rho ij}^{\alpha\beta kl} = \frac{1}{q} \delta_\gamma^\alpha \delta_\rho^\beta \delta_i^k \delta_j^l \delta_{ij} - \frac{1}{s} \delta_\gamma^\alpha \delta_\rho^\beta \delta_i^k \delta_j^l (\delta_{ij} - 1), \quad (3.6)$$

$$\mathcal{I}_{s\gamma\rho ii}^{\alpha\beta kl} \psi_k^\gamma \otimes \psi_l^\rho = \frac{1}{q} \psi_i^\alpha \otimes \psi_i^\beta, \quad \mathcal{I}_{s\gamma\rho ij}^{\alpha\beta kl} \psi_k^\gamma \otimes \psi_l^\rho = \frac{1}{s} \psi_i^\alpha \otimes \psi_j^\beta, \quad i \neq j,$$

and require that the  $\hat{R}$ -matrix satisfies the quantum Yang-Baxter equation

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}, \quad (3.7)$$

and the Hecke relation

$$\hat{R}^2 - (\hat{\mathcal{J}}_s - \hat{\mathcal{I}}_s) \hat{R} - \hat{J}_s = (\hat{J}_s + \hat{\mathcal{J}}_s \cdot \hat{R}) \cdot (\hat{J}_s - \hat{\mathcal{I}}_s \cdot \hat{R}) = 0, \quad (3.8)$$

where  $\hat{J}_s = J_{s\gamma\rho ij}^{\alpha\beta kl} = I_{\gamma\rho}^{\alpha\beta} I_{ij}^{kl}$  is the unit matrix,  $\hat{\mathcal{J}}_s$  is a diagonal matrix

$$\mathcal{J}_{s\gamma\rho ij}^{\alpha\beta kl} = q \delta_\gamma^\alpha \delta_\rho^\beta \delta_i^k \delta_j^l \delta_{ij} - s \delta_\gamma^\alpha \delta_\rho^\beta \delta_i^k \delta_j^l (\delta_{ij} - 1), \quad (3.9)$$

$$\mathcal{J}_{s\gamma\rho ii}^{\alpha\beta kl} \psi_k^\gamma \otimes \psi_l^\rho = q \psi_i^\alpha \otimes \psi_i^\beta, \quad \mathcal{J}_{s\gamma\rho ij}^{\alpha\beta kl} \psi_k^\gamma \otimes \psi_l^\rho = s \psi_i^\alpha \otimes \psi_j^\beta, \quad i \neq j,$$

and  $\hat{J}_s = \hat{\mathcal{I}}_s \cdot \hat{\mathcal{J}}_s$ .

It is not hard to check that the following anzats for  $R$ -matrix

$$\hat{R}_{\gamma\rho ij}^{\alpha\beta kl} = \hat{R}_{q\gamma\rho}^{\alpha\beta} \delta_i^k \delta_j^l \delta^{kl} + (s - s^{-1}) I_{\gamma\rho}^{\alpha\beta} \delta_i^k \delta_j^l \Theta^{kl} + \hat{Q}_{q\gamma\rho}^{\alpha\beta} \delta_j^k \delta_i^l \Theta^{kl} + (\hat{Q}_q^{-1})_{\gamma\rho}^{\alpha\beta} \delta_j^k \delta_i^l \Theta^{lk}, \quad (3.10)$$

where

$$\Theta^{kl} = \begin{cases} 1, & \text{if } k < l \\ 0, & \text{if } k \geq l \end{cases} \quad (3.11)$$

satisfies the Hecke equation (3.8).

In the product  $\Psi^{(b)} \otimes \Psi^{(b)}$  in (3.5) we imply the following ordering:

$$\Psi^{(b)} \otimes \Psi^{(b)} = \Psi_1^{(2)} \otimes \Psi_1^{(2)} \oplus \Psi_1^{(2)} \otimes \Psi_2^{(2)} \oplus \Psi_2^{(2)} \otimes \Psi_1^{(2)} \oplus \Psi_2^{(2)} \otimes \Psi_2^{(2)} \quad (3.12)$$

With this ordering the  $\hat{R}$ -matrix has the form:

$$\hat{R} = \begin{bmatrix} \hat{R}_q & 0 & 0 & 0 \\ 0 & (s - s^{-1}) \hat{I} & \hat{Q}_q & 0 \\ 0 & \hat{Q}_q^{-1} & 0 & 0 \\ 0 & 0 & 0 & \hat{R}_q \end{bmatrix}. \quad (3.13)$$

By analogy with (2.6) and (2.7), one can define from (3.8) the quantum  $(q, s)$ -anti-symmetrizer

$$\hat{\mathcal{A}}^{(q,s)} = (\hat{J}_s - \hat{\mathcal{I}}_s \cdot \hat{R}), \quad (3.14)$$

and the quantum  $(q, s)$ -symmetrizer

$$\hat{\mathcal{S}}^{(q,s)} = (\hat{J}_s + \hat{\mathcal{J}}_s \cdot \hat{R}) \quad (3.15)$$

which are orthogonal projectors

$$\begin{aligned}\hat{\mathcal{A}}^{(q,s)} \cdot \hat{\mathcal{S}}^{(q,s)} &= \hat{\mathcal{S}}^{(q,s)} \cdot \hat{\mathcal{A}}^{(q,s)} = 0, & (\hat{\mathcal{A}}^{(q,s)})^2 &= \hat{\mathcal{I}}_s^2 \cdot (\hat{\mathcal{I}}_s + \hat{\mathcal{J}}_s) \cdot \hat{\mathcal{A}}^{(q,s)}, \\ (\hat{\mathcal{S}}^{(q,s)})^2 &= \hat{\mathcal{J}}_s^2 \cdot (\hat{\mathcal{I}}_s + \hat{\mathcal{J}}_s) \cdot \hat{\mathcal{S}}^{(q,s)}.\end{aligned}$$

The quantum symmetrizer  $\hat{\mathcal{S}}^{(q,s)}$  allows one to define the commutation relations for the  $(q, s)$ -Grassmann bispinor  $\psi_i^\alpha$  ( $q^2 \neq -1$ )

$$\hat{\mathcal{S}}^{(q,s)} \cdot (\psi \otimes \psi) = 0, \quad \text{or} \quad \psi_i^\alpha \otimes \psi_j^\beta + (\hat{\mathcal{J}}_s \cdot \hat{R})_{\gamma\rho ij}^{\alpha\beta kl} \psi_k^\gamma \otimes \psi_l^\rho = 0. \quad (3.16)$$

Let us require that this commutation relations are covariant with respect to the transformation  $\psi$  by a quantum matrix  $\hat{A} \in SL_q(2, C)$ . Then taking into account (2.4), it is not hard to find the following solution of the quantum Yang-Baxter equation (3.7)

$$\hat{Q}_q = \hat{R}_q. \quad (3.17)$$

Substituting this solution in (3.16), we get

$$\begin{aligned}\psi_i^\alpha \otimes \psi_i^\beta &= -q \psi_i^\beta \otimes \psi_i^\alpha, & (\psi_i^\alpha)^2 &= 0, \\ \psi_2^\alpha \otimes \psi_1^\alpha &= -s q \psi_1^\alpha \otimes \psi_2^\alpha, \\ \psi_2^2 \otimes \psi_1^1 &= -s \psi_1^1 \otimes \psi_2^2, \\ \psi_2^1 \otimes \psi_1^2 &= -s \psi_1^2 \otimes \psi_2^1 - s(q - q^{-1}) \psi_1^1 \otimes \psi_2^2.\end{aligned} \quad (3.18)$$

To define the lattice  $(l, q, s)$ -Grassmann bispinor field  $\psi_i^\alpha(r)$  we consider the  $d$ -dimensional hypercubic lattice. The lattice sites are labeled by the vector  $r = (p_1, p_2, \dots, p_d)$ , where  $p_i$  are integer numbers and the total number of sites is  $N$ . For the sake of convenience, the lattice spacing  $a$  is fixed to be equal to unity. Let  $\psi_i^\alpha(r') \in \Psi_{r'}^{(b)}$  be the  $(q, s)$ -Grassmann bispinor on the site  $r'$ , which we consider as a "value" of the  $(l, q, s)$ -Grassmann field  $\psi_i^\alpha(r)$  on the site  $r'$ . The definition of the lattice  $(l, q, s)$ -Grassmann bispinor field requires some information about its commutation relation in two arbitrary sites  $r_m = (l_1, l_2, \dots, l_d)$  and  $r_n = (k_1, k_2, \dots, k_d)$ . In this paper we suppose the quasi-one-dimensional ordering for the lattice sites:  $r_m > r_n$ , if  $l_1 - k_1 > 0$  and the others  $l_i - k_i$  are arbitrary; if  $l_1 - k_1 = 0$ , then  $r_m > r_n$ , if  $l_2 - k_2 > 0$  and so on, and  $r_1 < r_2 < r_3 < \dots < r_N$  ( $m, n = 1, \dots, N$ ).

Let us denote a quantum vector space which is generated by  $4N$ -component  $(l, q, s)$ -Grassmann vector  $\psi_i^\alpha(r)$  as  $\Psi$  ( $N$  is the number of sites on the lattice). By analogy with (3.3), we must consider the quantum vector space  $\Psi$  as the Hecke sum with respect to some matrix  $\hat{Q}$

$$\Psi = \Psi_{r_1}^{(b)} \oplus_{\hat{Q}} \Psi_{r_2}^{(b)} \oplus_{\hat{Q}} \dots \oplus_{\hat{Q}} \Psi_{r_{N-1}}^{(b)} \oplus_{\hat{Q}} \Psi_{r_N}^{(b)}. \quad (3.19)$$

For definition of this sum and its  $\hat{\mathcal{R}}$ -matrix let us consider  $\hat{Q}$ -matrix as linear map  $\hat{Q} : \Psi_{r_n}^{(b)} \otimes \Psi_{r_m}^{(b)} \longrightarrow \Psi_{r_m}^{(b)} \otimes \Psi_{r_n}^{(b)}$

$$\hat{Q}(\psi_i^\alpha(r_n) \otimes \psi_j^\beta(r_m)) \equiv \hat{Q}_{\gamma\rho ij}^{\alpha\beta kl} \psi_k^\gamma(r_n) \otimes \psi_l^\rho(r_m) = -\frac{1}{l} \psi_i^\alpha(r_m) \otimes \psi_j^\beta(r_n), \quad (3.20)$$

$$(\hat{Q}^{-1})_{\gamma\rho ij}^{\alpha\beta kl} \psi_k^\gamma(r_m) \otimes \psi_l^\rho(r_n) = -l \psi_i^\alpha(r_n) \otimes \psi_j^\beta(r_m).$$

where  $r_m > r_n$  ( $m > n$ ).

By analogy with (3.6) we can define the  $\hat{\mathcal{R}}$ -matrix as linear map  $\hat{\mathcal{R}} : \Psi \otimes \Psi \longrightarrow \Psi \otimes \Psi$ :

$$\hat{\mathcal{R}}(\psi_i^\alpha(r) \otimes \psi_j^\beta(r')) \equiv \hat{\mathcal{R}}_{\gamma\rho ij}^{\alpha\beta kl}(r, r'|\tilde{r}, \tilde{r}') \psi_k^\gamma(\tilde{r}) \otimes \psi_l^\rho(\tilde{r}') = -\mathcal{I}_{\gamma\rho ij}^{\alpha\beta kl}(r, r'|\tilde{r}, \tilde{r}') \psi_k^\gamma(\tilde{r}) \otimes \psi_l^\rho(\tilde{r}'), \quad (3.21)$$

where we imply summation over values  $r_1, r_2, r_3, \dots, r_N$  of the discrete variables  $r, r', \tilde{r}, \tilde{r}'$  and  $\hat{\mathcal{I}}$  is the diagonal matrix

$$\mathcal{I}_{\gamma\rho ij}^{\alpha\beta kl}(r, r'|\tilde{r}, \tilde{r}') = \mathcal{I}_{s\gamma\rho ij}^{\alpha\beta kl} \delta(r, \tilde{r}) \delta(r', \tilde{r}') \delta(r, r') - \frac{1}{l} J_{s\gamma\rho ij}^{\alpha\beta kl} \delta(r, \tilde{r}) \delta(r', \tilde{r}') (\delta(r, r') - 1). \quad (3.22)$$

Let us require that the  $\hat{\mathcal{R}}$ -matrix satisfies the quantum Yang-Baxter equation

$$\hat{\mathcal{R}}_{12} \hat{\mathcal{R}}_{23} \hat{\mathcal{R}}_{12} = \hat{\mathcal{R}}_{23} \hat{\mathcal{R}}_{12} \hat{\mathcal{R}}_{23}, \quad (3.23)$$

and the Hecke relation

$$\hat{\mathcal{R}}^2 - (\hat{\mathcal{J}} - \hat{\mathcal{I}})\hat{\mathcal{R}} - \hat{\mathcal{J}} = (\hat{\mathcal{J}} + \hat{\mathcal{J}} \cdot \hat{\mathcal{R}}) \cdot (\hat{\mathcal{J}} - \hat{\mathcal{I}} \cdot \hat{\mathcal{R}}) = 0, \quad (3.24)$$

where  $\hat{\mathcal{J}} = J_{\gamma\rho ij}^{\alpha\beta kl}(r, r'|\tilde{r}, \tilde{r}') = J_{s\gamma\rho ij}^{\alpha\beta kl} \delta(r, \tilde{r}) \delta(r', \tilde{r}')$  is unit matrix,  $\hat{\mathcal{J}}$  is diagonal matrix

$$\mathcal{J}_{\gamma\rho ij}^{\alpha\beta kl}(r, r'|\tilde{r}, \tilde{r}') = \mathcal{J}_{s\gamma\rho ij}^{\alpha\beta kl} \delta(r, \tilde{r}) \delta(r', \tilde{r}') \delta(r, r') - l J_{s\gamma\rho ij}^{\alpha\beta kl} \delta(r, \tilde{r}) \delta(r', \tilde{r}') (\delta(r, r') - 1) \quad (3.25)$$

and  $\hat{\mathcal{J}} = \hat{\mathcal{I}} \cdot \hat{\mathcal{J}}$ .

The  $\hat{\mathcal{R}}$ -matrix satisfying the Hecke equation (3.24) has the following block structure:

$$\begin{aligned} \hat{\mathcal{R}}_{\gamma\rho ij}^{\alpha\beta kl}(r, r'|\tilde{r}, \tilde{r}') &= \hat{R}_{\gamma\rho ij}^{\alpha\beta kl} \delta(r, \tilde{r}) \delta(r', \tilde{r}') \delta(r, r') + (l - l^{-1}) J_{\gamma\rho ij}^{\alpha\beta kl} \delta(r, \tilde{r}) \delta(r', \tilde{r}') \Theta^{r r'} + \\ &\quad \hat{Q}_{\gamma\rho ij}^{\alpha\beta kl} \delta(r, \tilde{r}') \delta(r', \tilde{r}) \Theta^{r r'} + \hat{Q}_{\gamma\rho ij}^{\alpha\beta kl} \delta(r, \tilde{r}') \delta(r', \tilde{r}) \Theta^{r' r}, \end{aligned} \quad (3.26)$$

where the  $\hat{R}$ -matrix, and  $\Theta^{r r'}$  were defined in (3.10) and (3.11) respectively,  $\delta(r, \tilde{r})$  is a discrete  $\delta$ -function.

In the product  $\Psi \otimes \Psi$  in (3.21) we imply the following ordering:

$$\begin{aligned} \Psi \otimes \Psi &= \Psi_{r_1}^{(b)} \otimes \Psi_{r_1}^{(b)} \oplus \Psi_{r_1}^{(b)} \otimes \Psi_{r_2}^{(b)} \oplus \Psi_{r_2}^{(b)} \otimes \Psi_{r_1}^{(b)} \oplus \Psi_{r_2}^{(b)} \otimes \Psi_{r_2}^{(b)} \oplus \Psi_{r_2}^{(b)} \otimes \Psi_{r_1}^{(b)} \otimes \Psi_{r_3}^{(b)} \\ &\quad \oplus \Psi_{r_3}^{(b)} \otimes \Psi_{r_1}^{(b)} \oplus \Psi_{r_2}^{(b)} \otimes \Psi_{r_3}^{(b)} \oplus \Psi_{r_3}^{(b)} \otimes \Psi_{r_2}^{(b)} \oplus \Psi_{r_2}^{(b)} \otimes \Psi_{r_3}^{(b)} \otimes \Psi_{r_3}^{(b)} \oplus \dots \oplus \Psi_{r_N}^{(b)} \otimes \Psi_{r_N}^{(b)}. \end{aligned} \quad (3.27)$$

With this ordering the  $\hat{\mathcal{R}}$ -matrix has block structure and, for example, for two arbitrary sites  $r_n$  and  $r_m$  ( $r_n < r_m$ ) the block  $\tilde{\mathcal{R}}_{(nm)}$  of the  $\hat{\mathcal{R}}$ -matrix acting on the sum  $\Psi_{r_n}^{(b)} \otimes \Psi_{r_n}^{(b)} \oplus \Psi_{r_n}^{(b)} \otimes \Psi_{r_m}^{(b)} \oplus \Psi_{r_m}^{(b)} \otimes \Psi_{r_n}^{(b)} \oplus \Psi_{r_m}^{(b)} \otimes \Psi_{r_m}^{(b)}$  has the form:

$$\tilde{\mathcal{R}}_{(nm)} = \begin{bmatrix} \hat{R} & 0 & 0 & 0 \\ 0 & (l - l^{-1}) \hat{\mathcal{J}}_s & \hat{Q} & 0 \\ 0 & \hat{Q}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \hat{R} \end{bmatrix}. \quad (3.28)$$

By analogy with (3.14) and (3.15) from (3.24) we can define the quantum  $(l, q, s)$ -antisymmetrizer

$$\hat{\mathcal{A}} = (\hat{J} - \hat{\mathcal{I}} \cdot \hat{\mathcal{R}}), \quad (3.29)$$

and the quantum  $(l, q, s)$ -symmetrizer

$$\hat{\mathcal{S}} = (\hat{J} + \hat{\mathcal{J}} \cdot \hat{\mathcal{R}}) \quad (3.30)$$

which are orthogonal projectors

$$\hat{\mathcal{A}} \cdot \hat{\mathcal{S}} = \hat{\mathcal{S}} \cdot \hat{\mathcal{A}} = 0, \quad (\hat{\mathcal{A}})^2 = \hat{\mathcal{I}}^2 \cdot (\hat{\mathcal{I}} + \hat{\mathcal{J}}) \cdot \hat{\mathcal{A}}, \quad (\hat{\mathcal{S}})^2 = \hat{\mathcal{J}}^2 \cdot (\hat{\mathcal{I}} + \hat{\mathcal{J}}) \cdot \hat{\mathcal{S}}. \quad (3.31)$$

The quantum symmetrizer  $\hat{\mathcal{S}}$  permits to define the commutation relations for the lattice  $(l, q, s)$ -Grassmann bispinor field  $\psi_i^\alpha(r)$  ( $q^2 \neq -1$ )

$$\hat{\mathcal{S}} \cdot (\psi \otimes \psi) = 0, \quad \text{or} \quad \psi_i^\alpha(r) \otimes \psi_j^\beta(r') + (\hat{\mathcal{J}} \cdot \hat{\mathcal{R}})_{\gamma\rho ij}^{\alpha\beta kl}(r, r'|\tilde{r}, \tilde{r}') \psi_k^\gamma(\tilde{r}) \otimes \psi_l^\rho(\tilde{r}') = 0. \quad (3.32)$$

Let us require that these commutation relations are covariant with respect to the transformation of lattice field  $\psi_i^\alpha(r)$  by a quantum matrix  $\hat{A} \in SL_q(2, C)$  and  $\psi(r)$  is spinor with respect to  $SL_s(2, C)$ . These requirements lead to the following solution of the quantum Yang-Baxter equation (3.23)

$$\hat{Q} = \hat{Q}_q \hat{\otimes} \hat{Q}_s \quad \text{or} \quad \hat{Q}_{\gamma\rho ij}^{\alpha\beta kl} = \hat{Q}_{q\gamma\rho}^{\alpha\beta} \hat{Q}_{sij}^{kl}, \quad (3.33)$$

where  $\hat{Q}_q$  is defined in (3.17) and

$$\hat{Q}_s = \hat{R}_s = \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s - s^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s \end{bmatrix}.$$

Substituting this solution in (3.32), we obtain commutation relations

$$\begin{aligned} \psi_i^\alpha(r_m) \otimes \psi_j^\beta(r_m) &= -(\hat{\mathcal{J}}_s \cdot \hat{R})_{\gamma\rho ij}^{\alpha\beta kl} \psi_k^\gamma(r_m) \otimes \psi_l^\rho(r_m), \\ \psi_i^\alpha(r_m) \otimes \psi_i^\alpha(r_n) &= -l q s \psi_i^\alpha(r_n) \otimes \psi_i^\alpha(r_m), \\ \psi_i^\alpha(r_m) \otimes \psi_i^\beta(r_n) &= -l s \psi_i^\beta(r_n) \otimes \psi_i^\alpha(r_m), \\ \psi_i^\alpha(r_m) \otimes \psi_j^\alpha(r_n) &= -l q \psi_j^\alpha(r_n) \otimes \psi_i^\alpha(r_m), \\ \psi_i^\alpha(r_m) \otimes \psi_j^\beta(r_n) &= -l \psi_j^\beta(r_n) \otimes \psi_i^\alpha(r_m), \\ \psi_i^\beta(r_m) \otimes \psi_i^\alpha(r_n) &= -l s \psi_i^\alpha(r_n) \otimes \psi_i^\beta(r_m) - l s(q - q^{-1}) \psi_i^\alpha(r_n) \otimes \psi_i^\beta(r_m), \\ \psi_j^\alpha(r_m) \otimes \psi_i^\alpha(r_n) &= -l q \psi_i^\alpha(r_n) \otimes \psi_j^\alpha(r_m) - l q(s - s^{-1}) \psi_j^\alpha(r_n) \otimes \psi_i^\alpha(r_m), \\ \psi_j^\alpha(r_m) \otimes \psi_i^\beta(r_n) &= -l \psi_i^\beta(r_n) \otimes \psi_j^\alpha(r_m) - l(s - s^{-1}) \psi_j^\beta(r_n) \otimes \psi_i^\alpha(r_m), \\ \psi_i^\beta(r_m) \otimes \psi_j^\alpha(r_n) &= -l \psi_j^\alpha(r_n) \otimes \psi_i^\beta(r_m) - l(q - q^{-1}) \psi_j^\beta(r_n) \otimes \psi_i^\alpha(r_m), \end{aligned} \quad (3.34)$$

where  $r_n < r_m$ ,  $i > j$  and  $\alpha > \beta$ .

## 4 Integral calculus for the lattice $(l, q, s)$ -Grassmann field

At first let us briefly recall the definition of the  $GL_q(2, C)$ -covariant generalization of the Berezin integration over the quantum vector space  $\Psi^{(2)}$  [24].

Let  $\Psi^{(2)*}$  be space of the linear functional on  $\Psi^{(2)}$  with dual nilpotent basis  $\{\bar{d}\psi_\alpha\}$  ( $(\bar{d}\psi_\alpha)^2 = 0$ ) which maps  $\Psi^{(2)}$  in complex numbers:

$$(\Psi^{(2)*}, \Psi^{(2)}) \longrightarrow \mathbf{C}$$

and

$$(\bar{d}\psi_\alpha, \psi^\beta) = \int \bar{d}\psi_\alpha \psi^\beta = \delta_\alpha^\beta \quad (\bar{d}\psi_\alpha, c) = \int \bar{d}\psi_\alpha c = 0, \quad (4.1)$$

where  $c$  is a complex number.

Let us define the action of tensor product of the linear functionals by the following relation:

$$(\bar{d}\psi_\mu \otimes \bar{d}\psi_\nu, \psi^\gamma \otimes \psi^\rho) = (\bar{d}\psi_\mu, \psi^\gamma) \otimes (\bar{d}\psi_\nu, \psi^\rho) = \int \bar{d}\psi_\mu \otimes \bar{d}\psi_\nu \psi^\gamma \otimes \psi^\rho = \delta_\mu^\gamma \delta_\nu^\rho (\delta_{\mu\nu} - 1). \quad (4.2)$$

Using this definition and the following ansatz for commutation relations for  $\bar{d}\psi_\alpha$ :

$$\bar{d}\psi_{\alpha_1} \otimes \bar{d}\psi_{\beta_1} = x \bar{d}\psi_{\beta_1} \otimes \bar{d}\psi_{\alpha_1},$$

where  $\alpha_1 > \beta_1$ , we obtain (no summing)

$$\begin{aligned} \int \bar{d}\psi_{\alpha_1} \otimes \bar{d}\psi_{\beta_1} \psi^{\alpha_1} \otimes \psi^{\beta_1} &= 1 = -q \int \bar{d}\psi_{\alpha_1} \otimes \bar{d}\psi_{\beta_1} \psi^{\beta_1} \otimes \psi^{\alpha_1} = \\ &= -qx \int \bar{d}\psi_{\beta_1} \otimes \bar{d}\psi_{\alpha_1} \psi^{\beta_1} \otimes \psi^{\alpha_1} = -qx. \end{aligned}$$

From here  $x = -\frac{1}{q}$  and

$$\bar{d}\psi_{\alpha_1} \otimes \bar{d}\psi_{\beta_1} = -\frac{1}{q} \bar{d}\psi_{\beta_1} \otimes \bar{d}\psi_{\alpha_1},$$

or in covariant form

$$\bar{d}\psi_\alpha \otimes \bar{d}\psi_\beta + q (\hat{R}'_q)_{\alpha\beta}^{\gamma\rho} \bar{d}\psi_\gamma \otimes \bar{d}\psi_\rho = 0, \quad (4.3)$$

where  $(\hat{R}'_q)_{\alpha\beta}^{\gamma\rho} = \hat{R}_{q\beta\alpha}^{\rho\gamma}$  determines commutation relations for matrix elements of inverse matrix  $\hat{A}^{-1} \in GL_q(2, C)$

$$(\hat{R}'_q)_{\gamma\rho}^{\alpha\beta} (A^{-1})_\mu^\gamma (A^{-1})_\nu^\rho = (A^{-1})_\gamma^\alpha (A^{-1})_\rho^\beta (\hat{R}'_q)_{\mu\nu}^{\gamma\rho}. \quad (4.4)$$

Hence, it follows:

$$\text{if } \tilde{\psi}_i^\alpha = A_\beta^\alpha \otimes \psi^\beta, \quad \text{then } \bar{d}\tilde{\psi}_\alpha = (A^{-1})_\beta^\alpha \otimes \bar{d}\psi_\beta. \quad (4.5)$$

Using (4.1) and (4.2), one can define the  $GL_q(2, C)$ -invariant Berezin integral on  $\Psi^{(2)}$ :

$$\int \bar{d}\psi_1 \otimes \bar{d}\psi_2 \psi^1 \otimes \psi^2 = 1. \quad (4.6)$$

It is not hard to show that this integral is invariant with respect to transformations (4.5):

$$\int \bar{d}\tilde{\psi}_1 \otimes \bar{d}\tilde{\psi}_2 \tilde{\psi}^1 \otimes \tilde{\psi}^2 = \det_q(\hat{A}^{-1}) \det_q \hat{A} \int \bar{d}\psi_1 \otimes \bar{d}\psi_2 \psi^1 \otimes \psi^2 = 1. \quad (4.7)$$

Using (4.2) and (4.6), one can calculate the integral

$$\int \bar{d}\psi_1 \bar{d}\psi_2 \psi^\alpha \psi^\beta = \varepsilon^{\alpha\beta}, \quad (4.8)$$

where

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}. \quad (4.9)$$

and for brevity the notation of tensor product is omitted.

Now let us consider the integral calculus for the lattice  $(l, q, s)$ -Grassmann bispinor field  $\psi = \{\psi_i^\alpha(r)\}$  which satisfy commutation relations (3.34). By analogy with (4.1) let us define a space  $\Psi$  of the linear functional on  $\Psi$  (3.19) with dual nilpotent basis  $\{\bar{d}\psi_\alpha^i(r)\}$  ( $[\bar{d}\psi_\alpha^i(r)]^2 = 0$ ):

$$(\bar{\Psi}^*, \Psi) \longrightarrow \mathbf{C}$$

and

$$(\bar{d}\psi_\alpha^i(r), \psi_j^\beta(r')) = \int \bar{d}\psi_\alpha^i(r) \psi_j^\beta(r') = \delta_\alpha^\beta \delta_j^i \delta(r, r') \quad (\bar{d}\psi_\alpha^i(r), c) = \int \bar{d}\psi_\alpha^i(r) c = 0, \quad (4.10)$$

where  $c$  is a complex number.

Elements of the dual basis satisfy commutation relations

$$\bar{d}\psi_i^\alpha(r) \otimes \bar{d}\psi_j^\beta(r') + (\hat{\mathcal{J}} \cdot \hat{\mathcal{R}}')_{\gamma\rho ij}^{\alpha\beta kl}(r, r'|\tilde{r}, \tilde{r}') \bar{d}\psi_k^\gamma(\tilde{r}) \otimes \bar{d}\psi_l^\rho(\tilde{r}') = 0, \quad (4.11)$$

where  $\hat{\mathcal{R}}'$ -matrix has blok structure (3.28) and contains matrices  $(\hat{R}'_q)^{\gamma\rho}_{\alpha\beta} = \hat{R}_q^{\rho\gamma}_{\beta\alpha}$  and  $(\hat{R}'_s)^{kl}_{ij} = \hat{R}_s^{lk}_{ji}$ .

Let us define the  $GL_q(2, C)$ -covariant integration measure  $\mathcal{D}\psi$  for Berezin integral over the quantum vector space  $\Psi$ :

$$\begin{aligned} \mathcal{D}\psi &= \bar{d}\psi_1^1(r_1) \bar{d}\psi_2^1(r_1) \bar{d}\psi_1^2(r_1) \bar{d}\psi_2^2(r_1) \bar{d}\psi_1^1(r_2) \bar{d}\psi_2^1(r_2) \bar{d}\psi_1^2(r_2) \bar{d}\psi_2^2(r_2) \dots \\ &\quad \bar{d}\psi_1^1(r_N) \bar{d}\psi_2^1(r_N) \bar{d}\psi_1^2(r_N) \bar{d}\psi_2^2(r_N). \end{aligned} \quad (4.12)$$

By analogy with (4.8) and (4.9) we can define  $GL_q(2, C)$ -invariant generalization of the Berezin integration for the lattice  $(l, q, s)$ -Grassmann field

$$\begin{aligned} \int \mathcal{D}\psi \quad &\psi_{i_1}^{\alpha_1}(r_1) \psi_{i_2}^{\alpha_2}(r_1) \psi_{i_3}^{\alpha_3}(r_1) \psi_{i_4}^{\alpha_4}(r_1) \dots \psi_{j_1}^{\delta_1}(r_N) \psi_{j_2}^{\delta_2}(r_N) \psi_{j_3}^{\delta_3}(r_N) \psi_{j_4}^{\delta_4}(r_N) = \\ &\varepsilon_{i_1 i_2 i_3 i_4 \dots j_1 j_2 j_3 j_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \dots \delta_1 \delta_2 \delta_3 \delta_4}(r_1, \dots, r_N). \end{aligned} \quad (4.13)$$

Here the tensor  $\hat{\varepsilon}$  is defined by the following rules:

$$\varepsilon_{1122\dots 1122}^{1212\dots 1212}(r_1, \dots, r_N) = 1,$$

and the remaining components are defined by the coefficients appearing in the l.h.s of the relation

$$\begin{aligned} & \psi_{i_1}^{\alpha_1}(r_1) \psi_{i_2}^{\alpha_2}(r_1) \psi_{i_3}^{\alpha_3}(r_1) \psi_{i_4}^{\alpha_4}(r_1) \dots \psi_{j_1}^{\delta_1}(r_N) \psi_{j_2}^{\delta_2}(r_N) \psi_{j_3}^{\delta_3}(r_N) \psi_{j_4}^{\delta_4}(r_N) = \\ & \varepsilon_{i_1 i_2 i_3 i_4 \dots j_1 j_2 j_3 j_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \dots \delta_1 \delta_2 \delta_3 \delta_4}(r_1, \dots, r_N) \psi_1^1(r_1) \psi_1^2(r_1) \psi_2^1(r_1) \psi_2^2(r_1) \dots \psi_1^1(r_N) \psi_1^2(r_N) \psi_2^1(r_N) \psi_2^2(r_N) \end{aligned}$$

after reordering the l.h.s. to the r.h.s. using commutation relations (3.34) for each specific set of values of the indices.

Now let us consider the one-site Gaussian integral over lattice  $(l, q, s)$ -Grassmann field. In order to calculate this integral, one uses the following definition of  $(q, s)$ -Pfaffian (which is an extension of the definition of usual Pfaffian proposed in [27] to  $q$ -deformed commutation relations).

Consider the quadratic form

$$\begin{aligned} w = & a_{11}^{12} \psi_1^1 \psi_1^2 + a_{12}^{11} \psi_1^1 \psi_2^1 + a_{12}^{12} \psi_1^1 \psi_2^2 + a_{12}^{21} \psi_1^2 \psi_2^1 + \\ & a_{12}^{22} \psi_1^2 \psi_2^1 + a_{22}^{12} \psi_2^1 \psi_2^2 = \frac{1}{2} \sum_{\alpha, \beta, i, k} a_{ik}^{\alpha\beta} \psi_i^\alpha \psi_k^\beta, \end{aligned}$$

where the matrix elements  $a_{ik}^{\alpha\beta}$  are chosen to be commutative and the matrix  $\hat{a}$  has the form

$$\hat{a} = \begin{bmatrix} 0 & a_{11}^{12} & a_{11}^{11} & a_{12}^{12} \\ -\frac{1}{q} a_{11}^{12} & 0 & a_{12}^{21} & a_{12}^{22} \\ -\frac{1}{sq} a_{11}^{11} & -\frac{1}{s} a_{12}^{21} & 0 & a_{22}^{12} \\ -\frac{1}{s} a_{12}^{12} - (q - q^{-1}) a_{12}^{21} & -\frac{1}{sq} a_{12}^{22} & -\frac{1}{q} a_{22}^{12} & 0 \end{bmatrix}.$$

Here the lower-triangle matrix elements are determined using relations (3.18). Let us define  $(q, s)$ -Pfaffian of  $\hat{a}$  through

$$\frac{1}{2} w^2 = Pf_{(q,s)}(\hat{a}) \psi_1^1 \psi_1^2 \psi_2^1 \psi_2^2.$$

Then, taking into account commutation relations (3.18), we get

$$Pf_{(q,s)}(\hat{a}) = \frac{1}{2} (1 + q^2 s^4) a_{11}^{12} a_{22}^{12} - \frac{s}{2} (1 + q^2) a_{12}^{11} a_{12}^{22} + s q^2 a_{12}^{12} a_{12}^{21}. \quad (4.14)$$

Using (4.13) for the one site integral, it is immediate to show that

$$\int \bar{d}\psi_1^1 \bar{d}\psi_2^1 \bar{d}\psi_1^2 \bar{d}\psi_2^2 \exp \left\{ \frac{1}{2} \sum_{\alpha, \beta, i, k} a_{ik}^{\alpha\beta} \psi_i^\alpha \psi_k^\beta \right\} = Pf_{(q,s)}(\hat{a}). \quad (4.15)$$

Now let us show that, at  $q = \pm 1, s = \pm 1$ ,

$$Pf_{(q,s)}(\hat{a}) = \sqrt{\det(\hat{b})}, \quad (4.16)$$



where  $\det$  is the usual ( $q = 1$ ) determinant. To determine the matrix  $\hat{b}$  we consider  $(q, s)$ -bispinor  $v$ , which satisfies commutation relations

$$v_i^\alpha v_j^\beta = \frac{1}{q} \hat{R}_{\gamma\rho ij}^{\alpha\beta kl} v_k^\gamma v_l^\rho,$$

and require

$$(v_1^1)^2 = (v_1^2)^2 = (v_2^1)^2 = (v_2^2)^2 = 1 \quad \text{and} \quad [v_k^\alpha, \psi_l^\beta] = 0.$$

In order to define these commutation relations we use quantum  $(q, s)$ -antisymmetrizer (3.15). Then, at  $q = \pm 1, s = \pm 1$ ,  $\tilde{\psi}_i^\alpha = v_i^\alpha \psi_i^\alpha$  are the components of usual Grassmann bispinor  $\tilde{\psi}$ . Making use of the substitution  $\psi_i^\alpha \rightarrow \tilde{\psi}_i^\alpha = v_i^\alpha \psi_i^\alpha$  into integral (4.15), one gets

$$v_2^2 v_2^1 v_1^2 v_1^1 \int \bar{d}\tilde{\psi}_1^1 \bar{d}\tilde{\psi}_2^1 \bar{d}\tilde{\psi}_1^2 \bar{d}\tilde{\psi}_2^2 \exp \left\{ \frac{1}{2} \sum_{\alpha, \beta, i, k} b_{ik}^{\alpha\beta} \tilde{\psi}_i^\alpha \tilde{\psi}_k^\beta \right\} = Pf_{(q,s)}(\hat{a}).$$

where  $b_{ik}^{\alpha\beta} = a_{ik}^{\alpha\beta} v_i^\alpha v_k^\beta$  (here summation is absent).

Calculating this integral, we obtain the connection between the usual Pfaffian of the matrix  $\hat{b}$  and  $(q, s)$ -Pfaffian (4.14)

$$v_2^2 v_2^1 v_1^2 v_1^1 Pf(\hat{b}) = Pf_{(q,s)}(\hat{a}).$$

This relation yields

$$(Pf_{(q,s)}(\hat{a}))^2 = (Pf(\hat{b}))^2 = \det(\hat{b}).$$

Hence, we obtain (4.16).

The  $(l, q)$ -fermionization of the two-dimensional Ising model might be done using the lattice  $(l, q, s)$ -Grassmann field at  $q = l = -1, s = 1$  (see the following section). In what follows, we just consider this case, although to compare with the case  $l = q = 1$ , one sometimes has not to fix the value of  $l, q$ .

Note that for  $q = l = -1, s = 1$  the  $(l, q)$ -Grassmann field  $\psi_i^\alpha(r)$  is real. To show this, we define the real field as  $\bar{\psi}_i^\alpha(r) = \psi_i^\alpha(r)$ , where the bar denotes the Hermitian conjugation. Thus, we have

$$\overline{(\psi_i^\alpha(r) \psi_j^\beta(r'))} = \bar{\psi}_j^\beta(r') \bar{\psi}_i^\alpha(r) = \psi_j^\beta(r') \psi_i^\alpha(r).$$

On the other hand, the Hermitian conjugation of fifth relation in (3.34) gives the relation

$$\psi_i^2(r) \psi_j^1(r') = -\frac{1}{l_*} \psi_j^1(r') \psi_i^2(r).$$

The consistency condition for these relations leads to the restriction  $|l|^2 = 1$ . Using similarly arguments for (3.15), we get  $|q|^2 = |s|^2 = 1$ .

In further calculations we simplify the notation of the lattice  $(l, q)$ -Grassmann bispinor field

$$\psi(r) = (\psi_1^1(r), \psi_1^2(r), \psi_2^1(r), \psi_2^2(r)) \equiv (\psi_r^1, \psi_r^2, \psi_r^3, \psi_r^4),$$

and, as a corollary of (3.15) and (3.34) at  $l = q = -1, s = 1$ , its components satisfy the commutation relations

$$[\psi_r^1, \psi_r^2] = [\psi_r^3, \psi_r^4] = \{\psi_r^1, \psi_r^4\} = \{\psi_r^2, \psi_r^3\} = [\psi_r^1, \psi_r^3] = [\psi_r^2, \psi_r^4] = 0, \quad (4.17)$$

$$\begin{aligned}
[\psi_r^1, \psi_{r'}^2] &= [\psi_r^3, \psi_{r'}^4] = [\psi_r^1, \psi_{r'}^4] = [\psi_r^2, \psi_{r'}^3] = 0, \\
\{\psi_r^1, \psi_{r'}^3\} &= \{\psi_r^2, \psi_{r'}^4\} = \{\psi_r^\alpha, \psi_{r'}^\alpha\} = 0,
\end{aligned} \tag{4.18}$$

where  $\alpha = 1, 2, 3, 4$ .

Using commutation relations (4.17), (4.18) and (4.12)-(4.15), it is easy to calculate the one-site integral, which we need in the following section

$$\begin{aligned}
\int \bar{d}\psi_r \exp \left\{ \sum_{i < k} b_{ik} \psi_{r_i}^i \psi_{r_k}^k + b_4 \psi_{r_1}^1 \psi_{r_2}^2 \psi_{r_3}^3 \psi_{r_4}^4 \right\} \prod_{i=1}^4 \exp(\psi_{r_i}^i) = \\
= b_4 + Pf_{(l,q)}(b_{ik}) + b_{12} + b_{34} + b_{24} + b_{13} - b_{14} - b_{23},
\end{aligned} \tag{4.19}$$

where  $i, k = 1, 2, 3, 4$ ,  $l = q = -1$  and

$$\begin{aligned}
\int \bar{d}\psi_r &= \int \bar{d}\psi_{1r_1} \bar{d}\psi_{2r_2} \bar{d}\psi_{3r_3} \bar{d}\psi_{4r_4} = 0, \\
Pf_{(l,q)}(b_{ik}) &= b_{12} b_{34} + b_{13} b_{24} - b_{14} b_{23}.
\end{aligned} \tag{4.20}$$

## 5 $(l, q)$ -Fermionization of the two-dimensional Ising model

In this section we discuss how to express partition function of the two-dimensional Ising model through functional integral over the lattice  $(l, q)$ -fermion field ( $l = q = -1$ ). The method allows one to derive the Lagrangian of the lattice  $(l, q)$ -fermion field theory describing the behaviour of the Ising model nearby the critical point. Our method works for the two-dimensional Ising model of general type with the only restriction of  $Z_2$ -invariance of the plaquet statistical weight. In this case, the Hamiltonian  $h(\sigma_R)$  (energy per plaquet) includes not only the pairwise nearest-neighbour interactions, but also four-spin interaction

$$h(\sigma_R) = - \sum_{i < j} J_{ij} \sigma_R^i \sigma_R^j - J \prod_{i=1}^4 \sigma_R^i, \quad i, j = 1, \dots, 4, \tag{5.1}$$

where  $\sigma_R^i = \pm 1$  is the Ising spin,  $R = (x, y)$ ,  $x = 1, \dots, n$ ,  $y = 1, \dots, m$ , denote the plaquet position on the lattice (as before we use index  $r$  for numbering lattice sites),  $J_{ij}$  are the parameters of the pairwise interactions and  $J$  is the four-spin coupling constant (see Fig.)

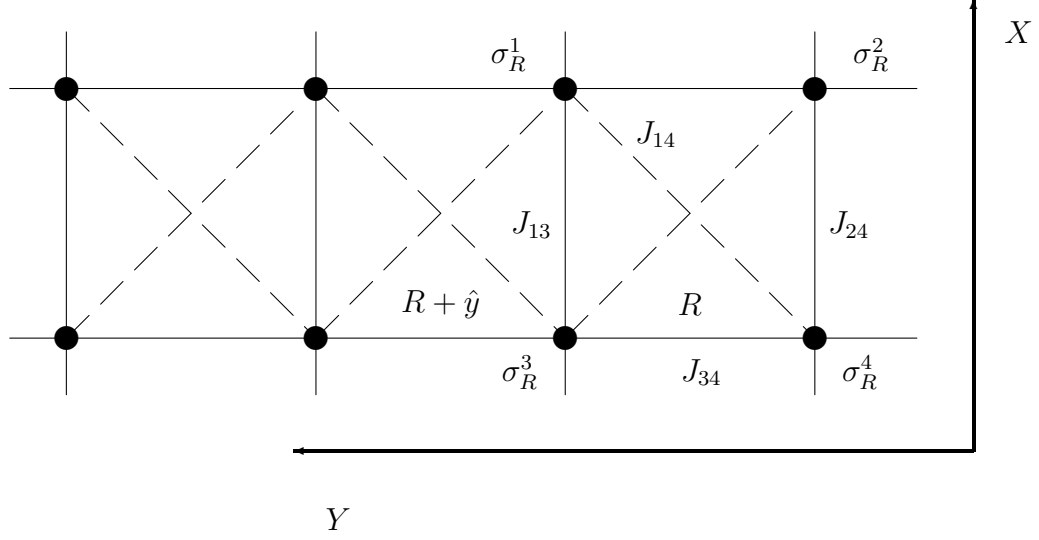
Note that for the Ising model with only the nearest-neighbours interactions we have

$$J_1 = J_{34}, \quad J_2 = J_{24}, \quad J = J_{12} = J_{13} = J_{24} = J_{14} = 0,$$

where  $J_1$  and  $J_2$  are the vertical and horizontal coupling constants, respectively.

The Ising model Hamiltonian on the whole lattice has the following form

$$H = \sum_R h(\sigma_R) = - \sum_R \left( \sum_{i < j} J_{ij} \sigma_R^i \sigma_R^j + J \prod_{i=1}^4 \sigma_R^i \right). \tag{5.2}$$



Here  $\hat{x}$  and  $\hat{y}$  denote the unit vectors along the horizontal  $X$  and vertical  $Y$  axes, respectively. This model corresponds to the eight-vertex model in external field [28], whose path integral obtained at the end of the section is not calculable due to the four-fermion interaction.

Partition function for the model with Hamiltonian (5.2) can be represented in the form of the product of plaquet statistical weights

$$Z = \sum_{\{\sigma\}} e^{-\beta H} = \sum_{\{\sigma\}} \prod_R \omega_R(\sigma_r, \sigma_{r+\hat{y}} | \sigma_{r+\hat{x}}, \sigma_{r+\hat{x}+\hat{y}}), \quad (5.3)$$

where  $\sigma_r$  is the Ising spin on site  $r$  and one gets at  $R = r$

$$\omega_R(\sigma_r, \sigma_{r+\hat{y}} | \sigma_{r+\hat{x}}, \sigma_{r+\hat{x}+\hat{y}}) = \omega_R(\sigma_R^1, \sigma_R^2 | \sigma_R^3, \sigma_R^4) = e^{-\beta h}. \quad (5.4)$$

It is easy to show that statistical weight (5.4) can be presented in the following form

$$\omega_R(\sigma) = \rho(1 + \sum_{i,k} \alpha_{ik} \sigma_R^i \sigma_R^k + \alpha_4 \prod_{i=1}^4 \sigma_R^i), \quad (5.5)$$

where

$$\alpha_{ik} = \langle \sigma_R^i \sigma_R^k \rangle_h, \quad \alpha_4 = \langle \sigma_R^1 \sigma_R^2 \sigma_R^3 \sigma_R^4 \rangle_h, \quad \rho = 2^{-4} \sum_{\sigma^1, \dots, \sigma^4} \omega_R(\sigma), \quad (5.6)$$

are expressed through the initial coupling constants  $J_{ik}$ ,  $J$  and  $\langle f(\sigma) \rangle_h$  denotes averaging over the plaquet statistical weight

$$\langle f(\sigma) \rangle_h = \sum_{\sigma^1, \dots, \sigma^4} f(\sigma) \omega_R(\sigma) / \sum_{\sigma^1, \dots, \sigma^4} \omega_R(\sigma).$$

Note that, choosing the statistical weight in the form

$$\omega_R(\sigma) = \exp(\mathcal{K}_1 \sigma_R^3 \sigma_R^4 + \mathcal{K}_2 \sigma_R^2 \sigma_R^4 + \mathcal{K} \prod_{i=1}^4 \sigma_R^i),$$

where  $\mathcal{K}_1 = \beta J_1$ ,  $\mathcal{K}_2 = \beta J_2$ ,  $\mathcal{K} = \beta J$ , one gets the eight-vertex model in zero external field, and, for the Ising model with only the nearest-neighbours interactions, one obtains

$$\omega_R(\sigma) = \exp(\mathcal{K}_1 \sigma_R^3 \sigma_R^4 + \mathcal{K}_2 \sigma_R^2 \sigma_R^4). \quad (5.7)$$

In the last case, using (5.6), it is easy to calculate the coefficients in (5.5)

$$\begin{aligned} \rho &= \cosh \mathcal{K}_1 \cdot \cosh \mathcal{K}_2, & \alpha_{34} &= t_2, & \alpha_{24} &= t_1, \\ \alpha_{23} &= t_1 t_2, & \alpha_{14} &= \alpha_{12} = \alpha_4 = 0, \end{aligned} \quad (5.8)$$

where  $t_1 = \tanh \mathcal{K}_1$  and  $t_2 = \tanh \mathcal{K}_2$ .

To obtain partition function (5.3) as a functional integral over the lattice  $(l, q)$ -fermion field, the key points are: commutation relations (4.17) and (4.18), definition of integration rule (4.13) over  $(l, q, s)$ -Grassmann field at  $l = q = -1, s = 1$  and representation (5.5) for the plaquet statistical weight. The latter one can be written using (4.19), (4.20) in the form of integral

$$\begin{aligned} \omega_R(\sigma_R^1, \sigma_R^2 | \sigma_R^3, \sigma_R^4) &= \rho \int \bar{d}\psi_R \exp \left[ \sum_{i < k} a_{ik} \sigma_R^i \sigma_R^k \psi_R^i \psi_R^k + \right. \\ &\quad \left. a_4 \sigma_R^1 \sigma_R^2 \sigma_R^3 \sigma_R^4 \psi_R^1 \psi_R^2 \psi_R^3 \psi_R^4 \right] \prod_{i=1}^4 \exp[\psi_R^i], \end{aligned} \quad (5.9)$$

where  $\bar{d}\psi_R = \bar{d}\psi_{1R} \bar{d}\psi_{2R} \bar{d}\psi_{3R} \bar{d}\psi_{4R}$ .

The parameters in (5.5) and (5.9) are expressed through each other as

$$\begin{aligned} \alpha_{12} &= a_{12}, \quad \alpha_{13} = a_{13}, \quad \alpha_{24} = a_{24}, \quad \alpha_{34} = a_{34}, \quad \alpha_{24} = a_{24}, \\ \alpha_{14} &= -a_{14}, \quad \alpha_{23} = -a_{23}, \quad \alpha_4 = a_4 + P f_{-1}(a_{ik}), \end{aligned}$$

and  $(l, q)$ -fermion field components  $\psi$  satisfy commutation relations (4.17) and (4.18).

In a similar way, one can write the representation of the statistical weight product in partition function (5.3)

$$\prod_R \omega_R(\sigma_r, \sigma_{r+\hat{y}} | \sigma_{r+\hat{x}}, \sigma_{r+\hat{x}+\hat{y}}) = \rho^{nm} \int \mathcal{D}\psi \exp \left[ \sum_r \mathcal{L}_r(\sigma\psi) \right] \prod_r \hat{\mathcal{P}}_r, \quad (5.10)$$

where

$$\begin{aligned} \mathcal{L}_r(\sigma\psi) &= \sigma_r \sigma_{r+\hat{y}} (a_{12} \psi_r^1 \psi_{r+\hat{y}}^2 + a_{34} \psi_r^3 \psi_{r+\hat{y}}^4) + \sigma_r \sigma_{r+\hat{x}} (a_{13} \psi_r^1 \psi_{r+\hat{x}}^3 + a_{24} \psi_r^2 \psi_{r+\hat{x}}^4) + \\ &\quad + \sigma_r \sigma_{r+\hat{x}+\hat{y}} a_{14} \psi_r^1 \psi_{r+\hat{x}+\hat{y}}^4 + \sigma_{r+\hat{x}} \sigma_{r+\hat{y}} a_{23} \psi_{r+\hat{y}}^2 \psi_{r+\hat{x}}^3 + \\ &\quad + a_4 (\sigma_r \sigma_{r+\hat{x}} \sigma_{r+\hat{y}} \sigma_{r+\hat{x}+\hat{y}}) (\psi_r^1 \psi_{r+\hat{y}}^2 \psi_{r+\hat{x}}^3 \psi_{r+\hat{x}+\hat{y}}^4), \\ \hat{\mathcal{P}}_r &= e^{\psi_r^1} e^{\psi_{r+\hat{y}}^2} e^{\psi_{r+\hat{x}}^3} e^{\psi_{r+\hat{x}+\hat{y}}^4}, \quad \mathcal{D}\psi = \prod_r (\bar{d}\psi_r). \end{aligned} \quad (5.11)$$

Since not all of the multipliers in  $\hat{\mathcal{P}}_r$  and  $\prod_r \hat{\mathcal{P}}_r$  are commutative, it is necessary to indicate their specific ordering. In (5.10) we imply the following ordering

$$\prod_r \hat{\mathcal{P}}_r = \prod_{y=1}^n \left( \prod_{x=1}^m \hat{\mathcal{P}}_{x,y} \right) = (\hat{\mathcal{P}}_{1,1} \cdot \hat{\mathcal{P}}_{2,1} \cdots \hat{\mathcal{P}}_{m,1}) \times$$

$$(\hat{\mathcal{P}}_{1,2} \cdot \hat{\mathcal{P}}_{2,2} \cdots \hat{\mathcal{P}}_{m,2}) \cdots (\hat{\mathcal{P}}_{1,n} \cdot \hat{\mathcal{P}}_{2,n} \cdots \hat{\mathcal{P}}_{m,n}). \quad (5.12)$$

Relation (5.10) is proved by termwise integration over the field components  $\psi_r^1, \psi_{r+\hat{y}}^2, \psi_{r+\hat{x}}^3, \psi_{r+\hat{x}+\hat{y}}^4$ .

Product (5.12) can be re-ordered in the following way

$$\prod_r \hat{\mathcal{P}}_r \implies \prod_r Q_r,$$

where

$$Q_r = e^{\psi_r^4} e^{\psi_r^3} e^{\psi_r^2} e^{\psi_r^1}.$$

Let us emphasize that, in the course of this re-ordering, we do not rearrange the non-commuting exponentials, for example,  $\exp(\psi_r^4)$  and  $\exp(\psi_{r'}^2)$  or  $\exp(\psi_r^3)$  and  $\exp(\psi_{r'}^1)$ . One can easily check it, writing down  $\hat{\mathcal{P}}_r$  in (5.12) explicitly through the corresponding exponentials in (5.11). Right in order to do this re-ordering, we need exotic commutation relations (4.17), (4.18) for the lattice  $(l, q)$ -fermion field at  $l = q = -1$ . Using this commutation relations and the nilpotence of the components  $\psi_r^i$ , one transforms  $Q_r$  to the form

$$Q_r = (1 + \sum_{i=1}^4 \psi_r^i) \exp[\mathcal{L}_r^{(0)}(\psi)],$$

where

$$\mathcal{L}_r^{(0)}(\psi) = \psi_r^4 \psi_r^3 + \psi_r^4 \psi_r^2 + \psi_r^4 \psi_r^1 + \psi_r^3 \psi_r^2 + \psi_r^3 \psi_r^1 + \psi_r^2 \psi_r^1.$$

As a result, one gets for integral (5.10)

$$\prod_R \omega_R = \rho^{nm} \int \mathcal{D}\psi \exp \left[ \sum_r \mathcal{L}_r(\sigma\psi) \right] \prod_r \left\{ (1 + \sum_{i=1}^4 \psi_r^i) \exp [\mathcal{L}_r^{(0)}(\psi)] \right\}. \quad (5.13)$$

The next obvious step is to replace the variables

$$\psi_r^i = \sigma_r \varphi_r^i,$$

which leads to vanishing the terms  $\sigma_r \sigma_{r'}$  in integral (5.10). Therefore, the dependence of the Ising spin is localized in the multiplier  $(1 + \sigma_r \sum_{i=1}^4 \varphi_r^i)$

$$1 + \sum_{i=1}^4 \psi_r^i = 1 + \sigma_r \sum_{i=1}^4 \varphi_r^i,$$

$$\mathcal{L}_r(\sigma\psi) = \mathcal{L}_r(\varphi), \quad \mathcal{L}_r^{(0)}(\psi) = \mathcal{L}_r^{(0)}(\varphi).$$

Now the summation over  $\sigma_r$  in partition function (5.3) is independently done at each lattice site

$$\sum_{\sigma_r=\pm 1} (1 + \sigma_r \sum_{i=1}^4 \varphi_r^i) = 2,$$

and the "bad" multiplier in product (5.13) vanishes. Since terms in  $\mathcal{L}_r^{(0)}(\varphi)$  and  $\mathcal{L}_r(\varphi)$  are quadratic over the fields  $\varphi_r$  (with an additional quartic term provided  $a_4 \neq 0$ ) and, therefore, commutative, one obtains

$$Z = (2\rho)^{nm} \int \mathcal{D}\varphi \exp \left[ \sum_r (\mathcal{L}_r^{(0)}(\varphi) + \mathcal{L}_r(\varphi)) \right], \quad (5.14)$$

where

$$\begin{aligned}\mathcal{L}_r^{(0)}(\varphi) &= \varphi_r^4 \varphi_r^3 + \varphi_r^4 \varphi_r^2 + \varphi_r^4 \varphi_r^1 + \varphi_r^3 \varphi_r^2 + \varphi_r^3 \varphi_r^1 + \varphi_r^2 \varphi_r^1, \\ \mathcal{L}_r(\varphi) &= a_{12} \varphi_r^1 \varphi_{r+\hat{y}}^2 + a_{34} \varphi_r^3 \varphi_{r+\hat{y}}^4 + a_{13} \varphi_r^1 \varphi_{r+\hat{x}}^3 + a_{24} \varphi_r^2 \varphi_{r+\hat{x}}^4 + \\ &\quad + a_{14} \varphi_r^1 \varphi_{r+\hat{x}+\hat{y}}^4 + a_{23} \varphi_r^2 \varphi_{r+\hat{x}+\hat{y}}^3 + a_4 \varphi_r^1 \varphi_{r+\hat{y}}^2 \varphi_{r+\hat{y}}^3 \varphi_{r+\hat{x}+\hat{y}}^4.\end{aligned}$$

At  $a_4 = 0$ , one can represent functional integral (5.14) as Gaussian integral

$$Z = (2\rho)^{nm} \int \mathcal{D}\varphi \exp \left[ \frac{1}{2} \sum \varphi_r^i D_{r,r'}^{ij} \varphi_{r'}^j \right], \quad (5.15)$$

where

$$\hat{D} = \begin{bmatrix} 0 & 1 + a_{12} \nabla_x & a_{13} - \nabla_{-x} & a_{14} \nabla_y + \nabla_{-x} \\ 1 + a_{12} \nabla_{-x} & 0 & a_{23} \nabla_{-y} + \nabla_{-x} & a_{24} - \nabla_x \\ a_{13} + \nabla_x & a_{23} \nabla_y + \nabla_x & 0 & 1 + a_{34} \nabla_y \\ a_{14} \nabla_{-y} + \nabla_x & a_{24} + \nabla_{-x} & 1 + a_{34} \nabla_{-y} & 0 \end{bmatrix}.$$

This integral is expressed through  $(l, q)$ -Pfaffian of the matrix  $\hat{D}$ . Similarly to the case of usual fermionization of the two-dimensional Ising model (1.2)-(1.4), this matrix is linear in the one-step shift operators  $\nabla_x, \nabla_y$ . In fact the accounting of the boundary condition in (5.15) gives the  $(l, q)$ -deformed counterparts ( $l = q = -1$ ) of (1.2), (1.3). For the sake of simplicity, we consider only  $Z^{AA}$ , where index "A" again denotes the antiperiodic boundary conditions along the axes  $X$  and  $Y$ . When turning to the momentum representation, the momentum components  $p_1, p_2$  take half-integer values.

In what follows, one restricts himself to considering the two-dimensional Ising model with only the nearest-neighbours interactions. In this case, the coefficients  $a_{ij}$  are written in (5.8) and functional integral (5.15) in momentum representation can be rewritten in the following form

$$Z = (2\rho)^{nm} \int \mathcal{D}\varphi \exp \left\{ \frac{1}{2} \sum_{p_1, p_2} \varphi^i(-p_1, -p_2) D^{ij}(p_1, p_2) \varphi^j(p_1, p_2) \right\}, \quad (5.16)$$

where

$$D^{ij}(p_1, p_2) = \begin{bmatrix} 0 & 1 & -e^{-ip_1} & e^{-ip_1} \\ 1 & 0 & e^{ip_1} - t_1 t_2 e^{-ip_2} & t_1 - e^{ip_1} \\ e^{ip_1} & e^{-ip_1} - t_1 t_2 e^{ip_2} & 0 & 1 + t_2 e^{ip_2} \\ e^{ip_1} & t_1 + e^{-ip_1} & 1 + t_2 e^{-ip_2} & 0 \end{bmatrix},$$

and

$$\begin{aligned}\mathcal{D}\varphi &= \prod_{p_i > 0} \bar{d}Re\varphi_1(p_1, p_2) \bar{d}Re\varphi_2(p_1, p_2) \bar{d}Re\varphi_3(p_1, p_2) \bar{d}Re\varphi_4(p_1, p_2) \\ &\quad \bar{d}Im\varphi_1(p_1, p_2) \bar{d}Im\varphi_2(p_1, p_2) \bar{d}Im\varphi_3(p_1, p_2) \bar{d}Im\varphi_4(p_1, p_2).\end{aligned}$$

At  $q = -1, s = 1$ , using (4.16) for the calculation of the  $(q, s)$ -deformed Gaussian functional integral, one immediately gets integral (5.16). As a result, one obtains known expression (1.6) for  $Z^{AA}$ .

Thus, (5.16) connects the partition function of the two-dimensional Ising model with functional integral in the lattice free  $(l, q)$ -fermion field theory ( $l = q = -1$ ). The coincidence of the results of calculations (5.16) and (1.8) allows one to establish the validity of the definition of the Berezin integral for the lattice  $(l, q, s)$ -Grassmann field, proposed in (4.13).

Note that in paper [29] it was demonstrated the connection between paragrassmann algebras and the some representation spaces of quantum matrix group  $L_q(2, C)$  with  $q$  being a  $(p + 1)$ -root of unity where  $p$  is integer ( $\theta^{p+1} = 0$  for paragrassmann  $\theta$ ). In our case, using definition of the grassmann anyonic field given in [30], the commutation relations (4.17)-(4.18) for the lattice  $(l, q, s)$ -fermion field ( $l = q = -1, s = 1$ ) can be considered as definition of the lattice two-component paragrassmann anyonic field with  $l = e^{i\pi\nu}$ , ( $\nu = 1$ ) and  $p = 3$ .

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